A proposal for studying off-shell stability of vacuum geometries in string theory

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Abstract

We briefly review studies of off-shell stability of vacuum geometries in semi-classical gravity. We propose a study of off-shell stability of vacua in string theory by a distinct, though somewhat related approach — by studying their stability under suitable world-sheet sigma model renormalization group (RG) flows. Stability under RG flow is a mathematically well-posed and tractable problem in many cases, as we illustrate through examples. The advantage is that we can make definite predictions about late time behaviour and end-points of off-shell processes in string theory. This is a contribution to the proceedings of Theory Canada 4, CRM Montreal.

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I   Introduction

Studies of off-shell stability in the context of semiclassical gravity are not new. It has been observed that off-shell perturbations of classical solutions that make the spacetime action negative lead to instabilities (in the semiclassical approximation) in quantum gravity [1], [2]. These instabilities are expected to cause quantum tunnelling from one classical configuration to another through off-shell configurations. An example of these is given by the Euclidean Schwarzschild instanton. There are perturbations of this background that make the spacetime action negative. One such mode, which is non-normalizable is expected to cause tunnelling between the large Schwarzschild black hole and hot flat space.

A similar exercise can be done for gravity in a cavity which is kept at fixed temperature in contact with a heat bath [3], [4], [5]. In this case, the action has three saddle points, hot flat space, the small (Euclidean) Schwarzschild black hole and the large black hole. As has been discussed in [6], the small black hole has an off-shell perturbation that makes the action negative, and is therefore unstable. It is expected to mediate in a quantum tunnelling between the large black hole and hot flat space. Furthermore, what is interesting is the link between local thermodynamic instability of the Lorentzian black hole (e.g., the small Schwarzschild black hole) and the off-shell perturbative instability of its Euclidean counterpart [3], [7]. As Reall points out, this perturbative instability is related to a classical Gregory-Laflamme instability of the black brane whose dimensional reduction (and Wick rotation) gives the Euclidean Schwarzschild black hole. The generalization of this analysis to dimensionally reduced black branes in supergravity also follows by looking at a suitable action (possibly involving other fields as well) and seeing if there are off-shell perturbations of a classical solution that can make the action negative [7].

In this article, we discuss a natural notion of 'off-shell stability' in closed string theory that is somewhat related to but distinct from the above analysis. One would expect to study off-shell stability of a geometry in string theory by considering off-shell configurations in an appropriate string field theory (for a recent review of string field theory, see [8]). However, at present, the study of off-shell stability in closed string field theory is a very hard problem. We will therefore work in the low energy approximation, and use the world-sheet sigma model approach. Then we can view an on-shell geometry (a string vacuum) as arising as a fixed point of the renormalization group (RG) flow

of the sigma model. As is well known, the sigma model corresponding to this geometry is then a conformal field theory (CFT). In many examples, it is easy to consider generic relevant perturbations of this CFT that are also tachyonic. Relevant perturbations induce RG flows of the sigma model. A perturbation that is tachyonic is indicative of a genuine off-shell instability in string field theory, and it is expected that there will be tachyon condensation (an off-shell process). Tachyon condensation generically leads to a change in geometry (for a review, see [9]). Thus it is natural to ask whether the two processes induced by the relevant tachyonic perturbation are related; i.e., whether the off-shell geometries that solve the RG flow equations are the same as those involved in the geometry change due to tachyon condensation [9]. There is some evidence that RG flow approximates the geometry change due to tachyon condensation, with the endpoints being the same in many cases. An example is the orbifold flow $C/Z_n \to C$ expected from tachyon condensation [10], [11], [12] — there is an exact solution to RG flow with precisely the same endpoints [13], [11]. In open string theory, there is an even more striking example — the Sen conjecture for unstable D-branes has been verified both using string field theory and world-sheet RG flows (see [9] for a review). In fact, the idea of exploring the configuration space in string (field) theory by studying RG flows of world-sheet sigma models is very old and was proposed in the late 1980s by Banks, Martinec, Vafa and others [14], [15]. It is hoped that future progress in string field theory will lead to a rigorous result relating evolution of off-shell perturbations in the theory to RG flow solutions in a suitable approximation.

Motivated by the above discussion, we propose to investigate off-shell stability of closed string vacua by studying their stability under suitable RG flows. (In)stability under RG flow is expected to be indicative of (in)stability under off-shell perturbations in string field theory (at least for a class of off-shell perturbations, in light of the above evidence). The advantage of the RG flow approach is that in many cases, the question of stability under RG flow is a mathematically well-posed problem. In this article, we first introduce some simple RG flows of string theory. We then discuss stability under suitable RG flows for different geometries of interest in string theory, highlighting the stability techniques used. Finally we comment on the possible applications of these stability results.

II RG flows in closed string theory, linear and geometric stability under RG flow

The β functions of the closed string world-sheet sigma models are obtained as a perturbative expansion in powers of the square of the string length α'. Truncating this expansion to first order in α' is a valid approximation as long as the magnitude of curvature of the target space is much less than $1/\alpha'$. The simplest first-order RG flow is the *Ricci flow*, which is well-studied in mathematics and has been used successfully to prove the Poincaré conjecture. This is a flow for the metric of the target space, g_{ij}, with respect to the RG flow parameter t, and what is physically significant is the flow of geometries (i.e., of metrics mod diffeomorphisms). Ricci flow is not form-invariant under arbitrary t dependent diffeomorphisms, so all flows related to each other by diffeomorphisms generated by a vector field V are physically equivalent. We write the most general flow in this class, also called the *Ricci-de Turck flow* as

$$\frac{\partial g_{ij}}{\partial t} = -\alpha'(R_{ij} + \nabla_i V_j + \nabla_j V_i). \tag{II.1}$$

This is the RG flow for the metric when all other fields are set to zero. In the case when there is a non-zero dilaton Φ, we can write $V_i = \nabla_i \Phi$ in (II.1) to get the flow for the metric. The flow for the dilaton is then

$$\frac{\partial \Phi}{\partial t} = \frac{\alpha'}{2}(\Delta\Phi - 2|\nabla\Phi|^2). \tag{II.2}$$

In the presence of a nonzero B field, we have a more complicated set of coupled equations which can be found, for e.g., in [16].

Fixed points of the Ricci-de Turck flow are candidates for being string vacua. It is well-known that when the manifold is compact, the only fixed points of Ricci-de Turck flow are Ricci flat [17] (this need not be true for RG flows involving other fields as well, see, for instance [18]). On noncompact manifolds, non-Ricci flat fixed points can exist, an example being the Witten black hole [19],[20]. Given such a fixed point of (II.2), the question of stability of the fixed point under Ricci flow for small perturbations is mathematically well-posed and can be investigated.

On compact manifolds, the technique used to investigate stability by Cao, Hamilton and Ilmanen [21] applies some results derived by Perelman as part

of his proof of the Poincaré conjecture [22]. Briefly, there is a diffeomorphism-invariant functional (the λ functional) that is constant only at fixed points of geometry under Ricci flow and monotonically increasing otherwise. Now, let us consider a fixed point with metric g (for which we can evaluate this functional, we call this $\lambda(g)$) and perturb this fixed point at some initial value of RG parameter t_0. Denote the perturbed metric at t_0 by $g^p(t_0)$. Evaluate λ for $g^p(t_0)$, denoted by $\lambda(g^p(t_0))$. The sign of $\lambda(g^p(t_0)) - \lambda(g)$ can be used to check for stability of the fixed point — if this sign is positive, then since λ for the perturbed geometry monotonically increases, the perturbed geometry will not approach the fixed point; and if this sign is negative, it will. $\lambda(g^p(t_0)) - \lambda(g)$ is then evaluated in a linearized approximation assuming that $g^p(t_0)$ is a small perturbation of g (note that this is only required at initial $t = t_0$, not all along the flow). For Ricci-flat metrics, this sign is related to the sign of the spectrum of the Lichnerowicz laplacian on the manifold. It has been shown that with the exception of scaling and other flat directions associated to the moduli space of the n-torus, the n-torus is linearly stable under Ricci flow. This has also been proven by other methods (maximal regularity) in [23]. This can be generalized to RG flow with a B-field [24]. Similarly, the Kähler-Ricci-flat metric on K3 is linearly stable except for a centre manifold of flat directions (and scaling) [23], [25].

Another notion of stability discussed in [21] is that of *geometric stability*. Consider a geometry that is a solution to RG flow but not a fixed point. One can still analyze whether this solution is an *attractor* for nearby metrics (up to diffeomorphisms and scaling). From the point of view of sigma model quantum field theory, such special geometries are interesting. But what would be the use of such a stability result in string theory? At the very least, we can examine the sigma model corresponding to the limiting geometry as $t \to \infty$ of this special solution since it is the natural end-point of many nearby RG flows. Thus, given the conjecture that RG flows approximate off-shell processes in string theory, this end-point may give insights into the end-points of off-shell processes. The quantum field theory at the end-point need not be a conformal field theory — an example when it is not is that of S^n. In [21], it is shown that S^n is geometrically stable. Therefore, we can ask what the sigma model corresponding to S^n flows to as $t \to \infty$. It is well-known that S^n shrinks under Ricci flow [26], and when its curvature is of order $1/\alpha'$, sigma model perturbation theory is no longer reliable. For the case of S^2, it is known (both using techniques from integrable models and numerical simulations) what the limiting world-sheet quantum field theory

is — it is a free field theory of *massive* fields [27], [28].

There is also the possibility that an attractor solution to an RG flow may be a genuine fixed point of another RG flow in string theory, possibly with the addition of other fields. For example, hyperbolic 3-space H^3 (or the BTZ black hole, which is a quotient geometry of H^3) is not a fixed point of the flow (II.1), but is a fixed point of RG flow with a non-zero B field [29], [30]. In such a case, a geometric stability result under the flow (II.1) is expected to be indicative of a linear stability result under RG flow with a B field, at least with respect to metric perturbations. Therefore, the question of whether H^3 is geometrically stable under Ricci flow is worth investigating. In fact, we know that H^n (i.e., Euclidean anti-de Sitter space AdS_n) arises in many contexts in string theory. There are supergravity solutions of the form $AdS_p \times S^q$ for specific integers $p, q > 0$. There are world-sheet descriptions as well for these geometries, and it is expected that they are fixed points of some RG flow that includes the effects of the RR fields. However the β function for the world-sheet theory is not known. So if we want to investigate stability of, say, $AdS_p \times S^q$ (or the Wick-rotated geometry $H^p \times S^q$), then the best we can do is investigate its geometric stability under Ricci flow. As discussed, it is known that S^q is geometrically stable. So a result on geometric stability of H^n under Ricci flow would at least imply stability of $H^p \times S^q$ under a special class of metric perturbations and would be the first step towards a complete stability analysis of $H^p \times S^q$. We also wish to mention that the classical stability of $AdS^p \times S^q$ in supergravity has been well-studied [31].

In the next section, we review a result on geometric stability of H^n under Ricci flow that is derived in [33]. In general, investigation of linear or geometric stability for non-compact manifolds is a difficult problem. This is because the analogue of the λ functional for noncompact manifolds only exists in certain cases (with some conditions on sign of curvature [32]). So techniques such as that of Cao, Hamilton and Ilmanen cannot be used. We briefly mention some noncompact geometries of interest and known stability results for these geometries. We then highlight a new stability technique motivated by relativity [34] and review how it is used to analyze stability of H^n in [33].

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III Stability of noncompact geometries

We first discuss linear stability results for noncompact fixed points of (II.1). The most obvious example is flat space. In this case, with conditions on the curvature and its decay asymptotically, stability results are derived in [35]. A more recent result is found in [36], where the metric at initial t is required to be a perturbation of the flat metric on R^n, and as well is required to satisfy some asymptotic conditions.

There exist nontrivial fixed points to (II.1) with $V_j \neq 0$ [37], [38]. Such solutions are referred to as *steady Ricci solitons* in Ricci flow literature. Stability results for some steady solitons can be found in [39] (with suitable asymptotic decay conditions imposed on the perturbation). One example of a Ricci soliton well-studied in physics is the (Wick-rotated) Witten black hole [19], [20], given by the metric

$$ds^2 = dr^2 + \tanh^2 r d\theta^2, \tag{III.1}$$

with $V_i = \nabla_i \Phi$ and $\Phi = \log(\cosh r)$. The Witten black hole arises as the backward limit (in a specific sense) of a collapsing solution to first-order RG (i.e., Ricci) flow, the sausage model [27] (in mathematics literature, this is called the Rosenau solution, see [26] for details). This may indicate the instability of the Witten black hole under specific perturbations. Thus the stability of the Witten black hole remains an open question.

Finally, we discuss the geometric stability of H^n (Euclidean AdS_n) under Ricci flow [33]. We use this example to introduce a new technique motivated by recent work of Andersson and Moncrief [34] in relativity. We first outline the technique. Consider a fixed point of an RG flow — a manifold with metric g_{ij}. Now, consider a perturbation of this fixed point, with the perturbed geometry having metric $g_{ij}^p = g_{ij} + h_{ij}$, where h_{ij} is assumed small. Then we can work with the linearized RG flow for the perturbation h_{ij}. We now define the *Sobolev norm* of the perturbation $\| h \|_{k,2}$,

$$
\begin{aligned}
(\| h \|_{k,2})^2 &= \int_M |h_{ij}|^2 \, dV + \int_M |\nabla_{p_1} h_{ij}|^2 \, dV + ... \\
&\quad ... + \int_M |\nabla_{p_1} .... \nabla_{p_k} h_{ij}|^2 \, dV.
\end{aligned}
\tag{III.2}
$$

In the above expression, $|h_{ij}|^2$, for example, is $h^{ij} h_{ij}$, and indices are raised with respect to the background metric g. We assume that this norm is defined

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for our perturbation at initial t. Then, if we can prove that the Sobolev norm of the perturbation is decaying under the linearized flow, i.e., $\| h \|_{k,2} \to 0$ as $t \to \infty$, then in many cases, we can prove using certain *Sobolev inequalities* that the perturbation and its derivatives decay *pointwise* everywhere on the manifold. We would like to illustrate this by studying geometric stability of H^n under Ricci flow. Recall that H^n is not a fixed point of Ricci flow — it expands uniformly under the flow. We are interested in whether its perturbations decay and if the perturbed geometry approaches the expanding H^n geometry under Ricci flow. However, the technique we just outlined was for studying stability of a *fixed point*. This does not pose a problem, since solutions to the Ricci flow

$$\frac{\partial \tilde{g}_{ij}}{\partial \tilde{t}} = -\alpha' \tilde{R}_{ij} \tag{III.3}$$

are related to those of the flow

$$\frac{\partial g_{ij}}{\partial t} = -\alpha'[R_{ij} + cg_{ij}] \tag{III.4}$$

by the rescalings

$$\tilde{t} = \frac{1}{\alpha' c} \ e^{\alpha' ct},$$
$$\tilde{g}_{ij} = e^{\alpha' ct} g_{ij}. \tag{III.5}$$

Writing the Riemann tensor of H^n as $R_{ijkl} = -\frac{1}{a}(g_{ik}g_{jl} - g_{il}g_{jk})$ where $a > 0$ is a constant, $R_{ij} = -\frac{(n-1)}{a} g_{ij}$ and $R = -\frac{n(n-1)}{a}$ (a characterizes the magnitude of curvature). We then see that H^n is a fixed point of the flow (III.4) with $c = \frac{(n-1)}{a}$. It is easy to see from (III.5) that linear stability of H^n under the flow (III.4) implies its geometric stability under the Ricci flow (III.3). To prove linear stability under (III.4), we first consider the linearized flow of the perturbation h_{ij} of H^n, which (after choosing an appropriate gauge), we write as

$$\frac{\partial h_{ij}}{\partial t} = (\Lambda h)_{ij} \tag{III.6}$$

where $(\Lambda\ h)_{ij} = \frac{\alpha'}{2} [(\Delta_L h)_{ij} - 2ch_{ij}]$ and Δ_L is the Lichnerowicz Laplacian operator that also appeared in studies of off-shell stability in semiclassical

gravity [2]. We need to consider only transverse perturbations as the divergence part of a perturbation is 'pure gauge', similar to perturbation analysis in classical relativity. We then define the following 'energy' integrals

$$E^{(K)} = \int_M \left| (\Lambda^{(K)}h)_{ij} \right|^2 \, dV, \tag{III.7}$$

where the notation $(\Lambda^{(2)}h)_{ij} = (\Lambda\Lambda h)_{ij}$, for example (denote $(\Lambda^{(0)}h)_{ij} = h_{ij}$). With appropriate asymptotic conditions on the perturbation, we can then derive the following bound for $n > 2$:

$$\frac{dE^{(K)}}{dt} \leq -\alpha' \frac{(n-2)}{a} E^{(K)} \tag{III.8}$$

Then, we can show that this bound implies that (with the appropriate asymptotic conditions) the Sobolev norm of the perturbation decays as $t \to \infty$ (see [33] for the details). For $n = 2$, we only need to consider the flow of the trace of the perturbation. We can prove similarly that the Sobolev norm of the trace decays under the flow. Choosing the Poincaré unit ball metric for H^n, we have the following Sobolev inequality derived by Andersson [40], for $s > n/2$ (s is a nonnegative integer):

$$\parallel h \parallel_{C^k} \quad \leq C \parallel h \parallel_{k+s,\,2} . \tag{III.9}$$

$\parallel h \parallel_{C^k}$ refers to the C^k norm of the perturbation. Let $\alpha = (\alpha_1, ...\alpha_j)$ be the j-tuple of non-negative integers α_i. We call α a multi-index. $\nabla^\alpha = \nabla_1^{\alpha_1}...\nabla_j^{\alpha_j}$ is a differential operator of order $|\alpha| = \sum_{i=1}^j \alpha_i$. The C^k norm of u is defined as

$$\parallel u \parallel_{C^k} = \max_{0 \leq |\alpha| \leq k} \, \sup_{x \in M} |\nabla^\alpha h(x)|, \tag{III.10}$$

where, for instance, $|h(x)| = [h^{ij}(x)h_{ij}(x)]^{\frac{1}{2}}$ is the usual pointwise tensor norm defined with respect to the background hyperbolic metric. Then the fact that the Sobolev norm of the perturbation decays as $t \to \infty$ implies from (III.9) that the perturbation and its derivatives decay pointwise on the manifold. This proves the geometric stability of H^n under Ricci flow.

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[2] We use the sign convention in mathematics literature for Δ_L. This differs by a sign from the convention used by physicists.

This technique can be easily generalized to other RG flows. H^3 is the fixed point of RG flow with a B-field. Thus a natural extension would be to study the stability of H^3 under this RG flow. Another question is the stability of quotient geometries of H^n. The most interesting of these is the BTZ black hole which is obtained as a quotient of H^3. A stability result for the BTZ black hole can probably be derived, but the study of boundary terms and asymptotic conditions will differ from our example. At this stage, we are unable to study general perturbations of $H^p \times S^q$, since this geometry evolves under Ricci flow (the H^p part expands and S^q shrinks), and we cannot find a flow with this geometry as fixed point, such that its solutions are related to those of Ricci flow by rescalings, similar to (III.4). However, these examples show that stability analysis of geometries under RG flows is in many cases, a well-posed problem and offers a tractable method of studying off-shell stability in string theory.

Finally, we observe that both in the off-shell stability studies in semi-classical gravity and in stability studies under the simplest RG flows, the Lichnerowicz operator (particularly the sign of its spectrum) plays a crucial role. We could study the stability of the same Euclidean black hole and black brane geometries considered by Reall under suitable RG flows. It is likely that a geometry that is semiclassically unstable would also be unstable under RG flow since both notions of off-shell stability rely on the spectrum of the Lichnerowicz operator. The advantage of the RG flow approach is that we can follow the evolution of the perturbation to late RG times, and make a prediction about the endpoint geometry. In cases where a geometry is linearly stable, it may be possible to extend this to a notion of *nonlinear* stability under RG flow, analogous to recent nonlinear stability results (for compact hyperbolic space) in relativity [34]. Thus we hope that this program will offer insights into late time behaviour and endpoints of off-shell processes in string theory.

IV   Acknowledgements

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